http://ucalgary.ca/~sneeuw/lectures/eng0423/heightsys.pdf

Height Systems

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1 Height Systems

1.1 Raw levelled heights

Levelled heights are non-unique. Depending on the levelling path one ends up with different heights for the same point. They have no physical relevance.

$$\Delta l_{0P} = \sum_{i=1}^{n} l_i$$

with: Δl_{0P} = levelled height difference between geoid and point P

l_i = levelled height differences (increments)

1.2 Geopotential numbers

Geopotential numbers are the basic height information. They are unique, i.e. path-independent.

$$C_P = W_0 - W_P = \int_0^P \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{x} = \int_0^P g \mathrm{d}H$$

with: C_P = geopotential number of point P

W_0 = potential on geoid

W_P = potential in point P

$\boldsymbol{g}, g$ = vector and scalar gravity

1.3 Dynamic height

Dynamic heights H^{dyn} are just geopotential numbers, scaled into a metric unit. Differences with respect to raw levelled heights can become large, depending on how well γ_0 represents gravity in the levelled area.

$$H_P^{\mathrm{dyn}} = \frac{C_P}{\gamma_0}$$

with: γ_0 = constant normal gravity value, e.g. $\gamma(45°)$

1.4 Orthometric height

The orthometric height H_P is the length of the plumbline between surface point P and the geoid, loosely speaking the *height above the geoid.*

$$H_P = \frac{C_P}{\bar{g}}$$

with: $\bar{g}$ = mean gravity along plumbline between P and geoid.

Orthometric heights require knowledge or assumptions of g inside the Earth. Gravity will behave rather linearly between P and geoid. Thus the average would be close to the value at half height $(\frac{1}{2}H)$. Assuming a crustal density of $\rho = 2670\,\mathrm{kg/m^3}$ and a free-air gradient of $0.3086\,\mathrm{mGal/m}$, we get the so-called

Prey-reduction: $\bar{g} = g(\frac{1}{2}H) = g_P + 0.0424H$,

in which the coefficient of H, which is $(\mathrm{BO} + \frac{1}{2}\mathrm{FA})$, is in $\mathrm{mGal/m}$. The procedure is to remove a Bouguer plate of thickness $\frac{1}{2}H$, downward continue over a vertical distance $\frac{1}{2}H$ and restore the plate again. The orthometric heights that use this mean gravity are called Helmert heights:

$$H_P = \frac{C_P}{g_P + 0.0424H}$$

We see that H occurs left and right. But since the Prey gradient is small, the accuracy of H in the denominator doesn't matter. One could use the raw levelled height and—if necessary—do an iteration.

So three assumptions are used in deriving Helmert orthometric heights:

i) linear behaviour of gravity between surface and geoid,

ii) a constant crustal density of $\rho = 2670\,\mathrm{kg/m^3}$,

iii) a fixed free-air gradient of $0.3086\,\mathrm{mGal/m}$.

1.5 Normal heights

Normal heights H^{n} do not have this problem. They are defined without any assumption as:

$$H_P^{\mathrm{n}} = \frac{C_P}{\bar{\gamma}}$$

with: $\bar{\gamma}$ = mean normal gravity along normal plumbline between P and quasi-geoid.

Normal gravity can be calculated at any point without hypothesis. Again, the normal gravity behaviour is very linear. The mean gravity value along the normal plumbline can therefore be approximated by the normal gravity valued at the point which has height $\frac{1}{2}H^{\mathrm{n}}$ above the ellipsoid. Neglecting second order terms in height we get:

$$\bar{\gamma} = \gamma(\phi_P, \tfrac{1}{2}H_P^{\mathrm{n}}) = \gamma(\phi_P, 0) - 0.1543H_P^{\mathrm{n}}$$

again with the coefficient in $\mathrm{mGal/m}$. Thus we end up with

$$H_P^{\mathrm{n}} = \frac{C_P}{\gamma_{P_0} - 0.1543H^{\mathrm{n}}} ,$$

which may be solved by inserting raw levelling heights in the denominator or by iteration.

The normal height H_P^{n} is interpreted as the height of the surface point P above the quasi-geoid. Alternatively it is interpreted as the height of the telluroid above the ellipsoid. They are related to Molodenskii's gravity field theory.

2 Height computations and corrections

Discretization Levelling and gravimetry produce a discrete set of data. The integral $\int g\mathrm{d}H$ needs to be discretized, which introduces a small error:

$$C_{PQ} = C_Q - C_P = \int_P^Q g\mathrm{d}H \approx \sum_{i=1}^{n} g_i l_i$$

Instead of using the above equations for height calculations, it would be easier to be able to use levelled height differences and correct them in some way. For a given height system, we would have formulas of the type:

$$\Delta H_{PQ} = H_Q - H_P = \Delta l_{PQ} + \text{small correction term.}$$

2.1 Dynamic heights

$$\begin{aligned} \Delta H^{\text{dyn}}_{PQ} &= \Delta l_{PQ} + \text{DC}_{PQ} \\ \text{dynamic correction:} \quad \text{DC}_{PQ} &= \sum_i \frac{g_i - \gamma_0}{\gamma_0} l_i \end{aligned}$$

2.2 Orthometric heights

$$\begin{aligned} \Delta H_{PQ} &= \Delta l_{PQ} + \text{OC}_{PQ} \\ \text{orthometric correction:} \quad \text{OC}_{PQ} &= \text{DC}_{PQ} - \text{DC}_{Q_0Q} + \text{DC}_{P_0P} \\ &= \sum_i \frac{g_i - \gamma_0}{\gamma_0} l_i + \frac{\bar{g}_P - \gamma_0}{\gamma_0} H_P - \frac{\bar{g}_Q - \gamma_0}{\gamma_0} H_Q \end{aligned}$$

2.3 Normal heights

$$\begin{aligned} \Delta H^{\text{n}}_{PQ} &= \Delta l_{PQ} + \text{NC}_{PQ} \\ \text{normal correction:} \quad \text{NC}_{PQ} &= \sum_i \frac{g_i - \gamma_0}{\gamma_0} l_i + \frac{\bar{\gamma}_P - \gamma_0}{\gamma_0} H^{\text{n}}_P - \frac{\bar{\gamma}_Q - \gamma_0}{\gamma_0} H^{\text{n}}_Q \end{aligned}$$

In all these equations g_i is the actual gravity along the levelling line, γ_0 is the fixed normal gravity used in the dynamic heights and $\bar{g}_P$ and $\bar{\gamma}_P$ respectively denote the mean gravity and normal gravity along (normal) plumbline.

3 Normal vs. true heights

The above height systems are *true heights* in the sense that they make use of gravity information, together with levelling. The only approximation made is the discretization of $\int g \mathrm{d}H$. If gravity is not available, the best guess would be to use normal gravity along the levelling line. This can be done for all height systems. It would produce normal geopotential numbers C^*_P, normal dynamic heights $H^{\text{dyn}^*}_P$ and normal orthometric heights H^*_P, together with their corrections NDC, NOC.

As an example the normal orthometric height is defined as

$$\begin{aligned} \text{normal geopotential number} \quad C^*_P &= \int_0^P \gamma(\phi, h) \mathrm{d}H \approx \sum_i \gamma(\phi_i, h_i) l_i = \sum_i \gamma_i l_i \\ \text{normal orthometric height} \quad H^*_P &= \frac{C^*_P}{\bar{\gamma}_P} \\ \text{normal orth. height diff.} \quad H^*_{PQ} &= \Delta l_{PQ} + \text{NOC}_{PQ} \\ \text{normal orth. correction} \quad \text{NOC}_{PQ} &= \sum_i \frac{\gamma_i - \gamma_0}{\gamma_0} l_i + \frac{\bar{\gamma}_P - \gamma_0}{\gamma_0} H_P - \frac{\bar{\gamma}_Q - \gamma_0}{\gamma_0} H_Q \end{aligned}$$

After some manipulation the normal orthometric correction between two successive levelling points i and j is simplified to:

$$\begin{aligned} \text{normal orth. correction} \qquad \text{NOC}_{ij} &\approx -0.0053\bar{H}_{ij}\,\Delta\phi_{ij}\,\sin 2\bar{\phi}_{ij} \\ \text{with} \qquad \bar{H}_{ij} &= \tfrac{1}{2}(H_i + H_j) \qquad \text{from raw heights} \\ \Delta\phi_{ij} &= \phi_j - \phi_i \\ \bar{\phi}_{ij} &= \tfrac{1}{2}(\phi_i + \phi_j) \end{aligned}$$

4 Vertical reference systems in use

4.1 CVGS28

- Canadian Vertical Geodetic System 1928
- normal orthometric heights
- official Canadian height system
- overconstrained datum: 4 tide gauges at the East coast and 2 at the West coast have been assigned the height 0.
- system has become obsolete, a new system is underway

4.2 NAVD88

- North American Vertical Datum 1988
- Helmert orthometric heights
- official US height system

4.3 IGLD55

- International Great Lakes Datum 1955
- dynamic heights
- height system in use around Great Lakes and St. Lawrence river
- main purpose: water management, civil/hydraulic engineering