

Global Models

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1 What is a “Global Model”?

In *geodesy*, by a *global model* we mean a *global gravity field model* of the Earth, also called a *Global Geopotential Model*. It is a mathematical function which describes the gravity field of the Earth in the 3-dimensional space. The determination of the Earth’s global gravity field is one of the main tasks of geodesy: it serves as a reference for geodesy itself, and it provides important information about the Earth, its interior and its fluid envelope for all geosciences.

2 Gravitation Versus Gravity

According to Newton’s law of gravitation (Newton, 1687) the magnitude of the attracting force F between two point-shaped masses m_1 and m_2 is

$$F = G \frac{m_1 m_2}{l^2} \quad (1)$$

where l is the distance between the two masses and G is the gravitational constant. The vector of the attracting force of a body with the density ρ in the volume v acting upon a point-shaped sample mass m at the point P is given by the volume integral:

$$\vec{F}(P) = G m \int_v \frac{\rho(q)}{l^2(P, q)} \frac{\vec{l}(P, q)}{l(P, q)} dv(q) \quad (2)$$

where $\vec{l}(P, q)$ is the vector from the point P to the volume element dv at the point q . Hence, this force can be thought of as the sum of all the forces produced by the (infinitely many and infinitely small) mass elements ρdv in the volume v . The force \vec{F} can be divided by the sample

mass m to get the acceleration \vec{a} caused by the density distribution ρ :

$$\vec{a}(P) = G \int_v \frac{\rho(q)}{l^2(P, q)} \frac{\vec{l}(P, q)}{l(P, q)} dv(q) \quad (3)$$

If ρ and v are the density distribution and the volume of the Earth, respectively, the vector field \vec{a} is the gravitational field of the Earth.

Such a vector field can be described by a scalar field $V(P)$, a potential, in such a way that the vector field \vec{a} is the gradient of V :

$$\vec{a}(P) = \nabla V(P) \quad (4)$$

where ∇ is the Nabla operator (see e.g. Bronshtein et al, 2007). Since the gradient of a constant scalar field is zero, the potential V is uniquely determined from the vector field \vec{a} except for an unknown constant. In geodesy, this unknown constant is chosen in such a way that the potential V becomes zero in infinity. With this definition, the gravitational potential at point P describes the energy which is necessary to move a unit mass from point P against the attraction force field \vec{a} to infinity, i.e., the potential is always positive and decreases with distance to the mass distribution. As a consequence of Eq. 3, the gravitational potential of a mass distribution is given by the integral:

$$V(P) = G \int_v \frac{\rho(q)}{l(P, q)} dv(q) \quad (5)$$

and satisfies *Poisson's equation*:

$$\nabla^2 V(P) = -4\pi G \rho(P) \quad (6)$$

where ∇^2 is called the Laplace operator (see e.g. Bronshtein et al, 2007). Outside the masses the density ρ is zero and V satisfies *Laplace's equation*.

$$\nabla^2 V(P) = 0 \quad (7)$$

In mathematics, a function satisfying Laplace's equation is called a *harmonic function*. Any potential can be visualized by its *equipotential surfaces*, i.e. the surfaces where the potential has the same value. Moreover, from the theory of harmonic functions it is known that the knowledge of one equipotential surface is sufficient to define the whole harmonic function.

On a mass point rotating together with the Earth (e.g. lying on the Earth's surface or flying in the atmosphere), additionally to the gravitational attraction of the Earth, also the *centrifugal acceleration*

$$\vec{z}(P) = \omega^2 \vec{d}_z(P) \quad (8)$$

is acting due to the rotation of the Earth, where ω is the angular velocity of the Earth and $\vec{d}_z(P)$ is the shortest vector from the rotational axis to the point P (i.e. it is orthogonal to the rotational axis). The corresponding (non-harmonic) *centrifugal potential* (for which $\vec{z} = \nabla\Phi$) is:

$$\Phi(P) = \frac{1}{2} \omega^2 d_z^2(P) \quad (9)$$

Hence, in an Earth fixed (rotating) coordinate system the acceleration \vec{g} , called *gravity vector*, acting on a unit mass, is the sum of gravitational attraction and centrifugal acceleration (see Eqs. 3 and 8):

$$\vec{g}(P) = \vec{a}(P) + \vec{z}(P) \quad (10)$$

and the corresponding *gravity potential* is (see Eqs. 5 and 9):

$$W(P) = V(P) + \Phi(P) \quad (11)$$

The magnitude of the gravity vector is the well-known gravity

$$g(P) = |\vec{g}(P)| = |\vec{a}(P) + \vec{z}(P)| \quad (12)$$

Although the two being sometimes confused with each other, *gravity* and *gravitation* (*gravity potential* and *gravitational potential* respectively) are two different matters: gravity contains the centrifugal acceleration whereas gravitation does not.

As mentioned above, equipotential surfaces can be used to visualize a potential. In geodesy, one equipotential surface of the Earth's gravity potential is of particular importance: the *geoid*. Among all equipotential surfaces the geoid is the one which coincides with the undisturbed sea surface (i.e., sea in static equilibrium) and its fictitious continuation below the continents. The geoid is the natural physical height reference surface, i.e. the *height datum*. To define the surface of the geoid in space, the correct value W_0 of the potential has to be chosen:

$$W(P) = W_0 = \text{constant} \quad (13)$$

Textbooks for further study of physical geodesy are Heiskanen and Moritz (1967); Pick et al (1973); Vaníček and Krakiwsky (1982); Torge (1991).

3 Global Models

In geodesy, a mathematical function which approximates the real gravity potential of the Earth in the space outside the Earth is called a *global gravity field model* or simply a *global model*. From such an approximating gravity potential all related gravity field functionals (e.g., gravity potential, gravity vector, gravity) can be computed (see, e.g. Barthelmes, 2013). However, the gravity field functionals *geoid height*, *gravity anomaly* and *gravity disturbance*, which are of particular importance in geodesy, can only be computed with respect to a defined *reference system*, e.g., the Geodetic Reference System *GRS80* (Moritz, 1980). As the centrifugal part can be modeled easily and very accurately (see Eqs. 8 and 9) and is also part of the reference system, the challenging task is to model, i.e., to approximate, the gravitational potential. Thus, an appropriate mathematical representation, i.e., a set of basis functions, is needed which allows to approximate a harmonic function outside the Earth.

3.1 Mathematical Representation

Although there are many different mathematical representations for such a model (e.g., *ellipsoidal harmonics*, *spherical radial basis functions*, *spherical harmonic wavelets*), up to now, solid *spherical harmonics* are the basis functions used in practice almost exclusively. The solid

spherical harmonics are an orthogonal set of solutions of the Laplace equation (see Eq. 7) represented in a system of spherical coordinates (see, e.g. Hobson, 1931; Heiskanen and Moritz, 1967). Thus, the Earth's gravitational potential V at any point (r, λ, φ) on and above the Earth's surface can be expressed by summing up over degree and order of a spherical harmonic expansion as follows:

$$V(r, \lambda, \varphi) = \frac{GM}{r} \sum_{\ell=0}^{\ell_{\max}} \sum_{m=0}^{\ell} \left(\frac{R}{r}\right)^{\ell} P_{\ell m}(\sin \varphi) (C_{\ell m} \cos m\lambda + S_{\ell m} \sin m\lambda) \quad (14)$$

The notation is:

r, λ, φ	-	spherical geocentric coordinates of computation point (radius, longitude, latitude)
R	-	reference radius
GM	-	product of gravitational constant and mass of the Earth
ℓ, m	-	degree and order of spherical harmonic
$P_{\ell m}$	-	Legendre functions (normalized)
$C_{\ell m}, S_{\ell m}$	-	Stokes' coefficients (normalized)

and the Legendre functions are normalized in such a way that

$$\frac{1}{4\pi} \int_{\lambda=0}^{2\pi} \int_{\varphi=0}^{\pi} [P_{\ell m}(\sin \varphi) \cos m\lambda]^2 d\lambda d\varphi = 1 \quad (15)$$

With this representation, a global model is given by the parameters GM and R and the normalized coefficients $C_{\ell m}$ and $S_{\ell m}$ up to an upper limit ℓ_{\max} of degree ℓ . Such an expansion of a potential in solid spherical harmonics is a representation in the frequency domain, i.e. Eq. 14 relates the spatial and the spectral domains of the potential. The upper limit ℓ_{\max} of the summation governs the shortest wavelength representable by the model, i.e., the higher ℓ_{\max} , the higher the spatial resolution of the model. Hence, to improve such a model, the accuracy of the parameters $C_{\ell m}$ and $S_{\ell m}$ and/or the resolution, i.e., ℓ_{\max} , must be increased. To also model the centrifugal part, the parameter ω , the angular velocity of the Earth, must be known (see Eqs. 9 and 11). From this representation of the potential in spherical harmonics all other functionals of the gravity field can easily be derived. The gravity vector \vec{g} , for example, can be computed from Eq. 14 using Eqs. 4, 10 and 8 (see, e.g. Barthelmes, 2013).

3.2 History and Determination

How many parameters of a global gravity field model can be estimated and how to do this depend on what information (i.e., which measurements) is available. One can argue that with Newton's law of universal gravitation (Eq. 1) (Newton, 1687) also the first gravitational field model of the Earth was created. At that time, numerical values of the *free fall acceleration* on the Earth were known from *Galileo's* experiments, and also approximate values of the Earth's size had existed for a long time since the first estimates by *Eratosthenes* around 200 BC. Thus, approximations of the product GM , i.e., the Earth's mass M times gravitational constant G , can be calculated, and the spherically symmetric potential

$$V(r, \lambda, \varphi) = V(r) = \frac{GM}{r} \quad (16)$$

which remains from Eq. 14 (with $\ell_{max} = 0$ and $C_{0,0} = 1$), could be considered as the first gravitational field model. Up to now, the product GM could have been estimated much more accurately than the separation into the two factors M and G has been possible.

The next important improvement was the estimation of the *Earth's flattening* and its effect on the gravitational field, described by the parameter $C_{2,0}$ in Eq. 14, from geometrical and gravity measurements on the Earth and from the Lunar orbit. More information about the history of geodesy can be found, e.g., in Vaníček and Krakiwsky (1982) and Smith (1997).

Probably the most important step in the history of determining global gravity field models was the launch of the first artificial Earth's satellites. It heralded the start of a new era in geodesy, the *satellite geodesy* or *space geodesy*. Already in 1958, shortly after the launch of *Sputnik 2* in November 1957, the parameter $C_{2,0}$ could be computed with an unprecedented accuracy (Merson and King-Hele, 1958). In the following decades global gravity field models have been continuously improved by means of various artificial satellites orbiting the Earth. The basic principle is the following: The orbit of a satellite flying around a perfectly spherical planet with a homogeneous density distribution (or a density structure only dependent on the radius) would be a *Keplerian Ellipse*. The gravitational field of such a planet would be described by the simple potential of Eq. 16. Indeed, in a first rough approximation, the Earth is spherical and the orbits of satellites can, fairly reasonable, be approximated by Keplerian ellipses. However, the flattening of the Earth and the irregularly distributed masses inside and on the Earth, including the topography, cause a gravitational field with slightly ellipsoidal-shaped equipotential surfaces which, in addition, have “bumps” and “dents”. As a consequence, the real orbits are no ellipses. In turn, the measuring of these deviations from ellipsoidal orbits, the so called *orbit perturbations*, can be used to compute the flattening and the finer structures of the gravitational field. Using this method, the satellite itself is the sensor and its orbit must be measured accurately. For this purpose, at first photographs of satellite traces were taken against the star background. Later, being the next step to improve the gravity field models, the satellite orbits were measured by *laser ranging*. To reflect the laser pulses, the satellites are equipped with retroreflectors, the so called triple prisms. The two most important satellites, built especially for this purpose, are *Starlette* (launched 1975), and *LAGEOS* (launched 1976).

To compute continuously improving global gravity field models in the period before powerful computers were available, the linear analytical *orbit perturbation theory* of Kaula had played a very important role (Kaula, 1966). He had formulated the orbit perturbations in dependence of the spherical harmonic representation of the gravitational field, i.e., in relation to the coefficients $C_{\ell m}$ and $S_{\ell m}$ (Eq. 14). Nowadays, the parameters of a global gravity field model are usually computed from orbital data by numerical integration of the satellite orbits using an approximate model, where the parameters of this model can then be (iteratively) improved from the differences between the computed and observed orbits.

Important steps in improving the global gravity field models were the three satellite missions *CHAMP* (CHALLENGING Minisatellite Payload, in orbit from 2000 to 2010), *GRACE* (Gravity Recovery And Climate Experiment, launched 2002) and *GOCE* (Gravity Field and Steady-State Ocean Circulation Explorer, in orbit from 2009 to 2013). Similar to the previous satellites, CHAMP was used to compute the gravitational field by measuring its orbit. However, two things were new: the CHAMP orbit was measured continuously and very accurately using the high-flying satellites of the *Global Navigational Satellite System* (GNSS), and the measurements of a 3-axial accelerometer, positioned at the center of mass of CHAMP, for the first time, allowed

to separate the forces acting on the surface of the satellite (like drag and radiation pressure) from the gravitational forces. This means that the *satellite to satellite tracking* in a so called *high-low* configuration has been used. Alike CHAMP, the twin GRACE satellites are also equipped with GNSS receivers and accelerometers for tracking the orbits and measuring the non-gravitational effects. Furthermore, a microwave ranging system measures the change in distance between the tandem satellites very accurately and provides additional information about the gravitational field. This is a combination of a high-low configuration with the so-called *low-low* configuration.

To compute a global gravity field model in the years before CHAMP and GRACE, it was necessary to collect measurements to satellites over several years to get a stable solution. Thus, these models represent the gravity field averaged over the time period of the used data. It were CHAMP and in particular GRACE that enabled the computation of global gravity field models using measurements from much shorter periods like one month or one week only. For the first time, the variation of the Earth's global gravity field in space *and time* could be observed. This is the main reason why nowadays the modeling of the global gravity field is no longer a purely geodetic task, but is becoming more and more important for all Earth sciences dealing with mass transports at the corresponding time scale (oceanography, hydrology, glaciology, seismology, climatology, meteorology).

With GOCE, for the first time a gravity *gradiometer* was used aboard a satellite to gain information about the gravitational field. The orbit of GOCE was also determined by using the GNSS satellites, i.e., by high-low satellite-to-satellite tracking. The gradiometer measurements (i.e., the tensor of the second derivatives of the potential) are particularly sensitive to the shorter wavelengths of the gravitational field, thus, with GOCE the spatial resolution of the models could be further improved.

The global gravity field models are usually subdivided into *satellite-only models* and *combined models*. The satellite-only models are computed from satellite measurements alone, whereas for the combined models terrestrial gravity measurements over the continents and measurements of the mean sea surface from altimetry over the oceans are used additionally. The spatial resolution of the satellite-only models is lower because the shorter wavelengths of the gravity field are damped much stronger with increasing distance from the Earth (which is mathematically evident from the radius term $(R/r)^\ell$ of Eq. 14), and the orbits cannot be arbitrarily low (280 – 250 km for GOCE). On the other hand, the accuracy and the spatial resolution of these models are nearly uniform over the Earth (apart from possible polar gaps, depending on the inclination of the orbit), and they are not affected by possible errors in modeling the *sea surface topography*, which is necessary if altimetry over the oceans is used to compute a combined gravity field model.

The satellite-only models with the highest spatial resolution are the ones containing gradiometer measurements from GOCE. Their spherical harmonic coefficients can be computed up to a maximum degree and order of 250 – 300, which corresponds to smallest representable bumps and dents of ca. 100 – 80 km extent (half-wavelength resolution).

The best combined models have a maximum degree and order of ca. 2000 and a spatial resolution of about 10 km (half-wavelength). However, in practice this spatial resolution only exists in regions where dense and accurate terrestrial measurements are included in the model.

All global gravity field models which are available so far, from the first models to the most recent ones, are collected by the International Centre for Global Earth Models (*ICGEM*) (Barthelmes and Köhler, 2012) and can be downloaded or used for calculating gravity field

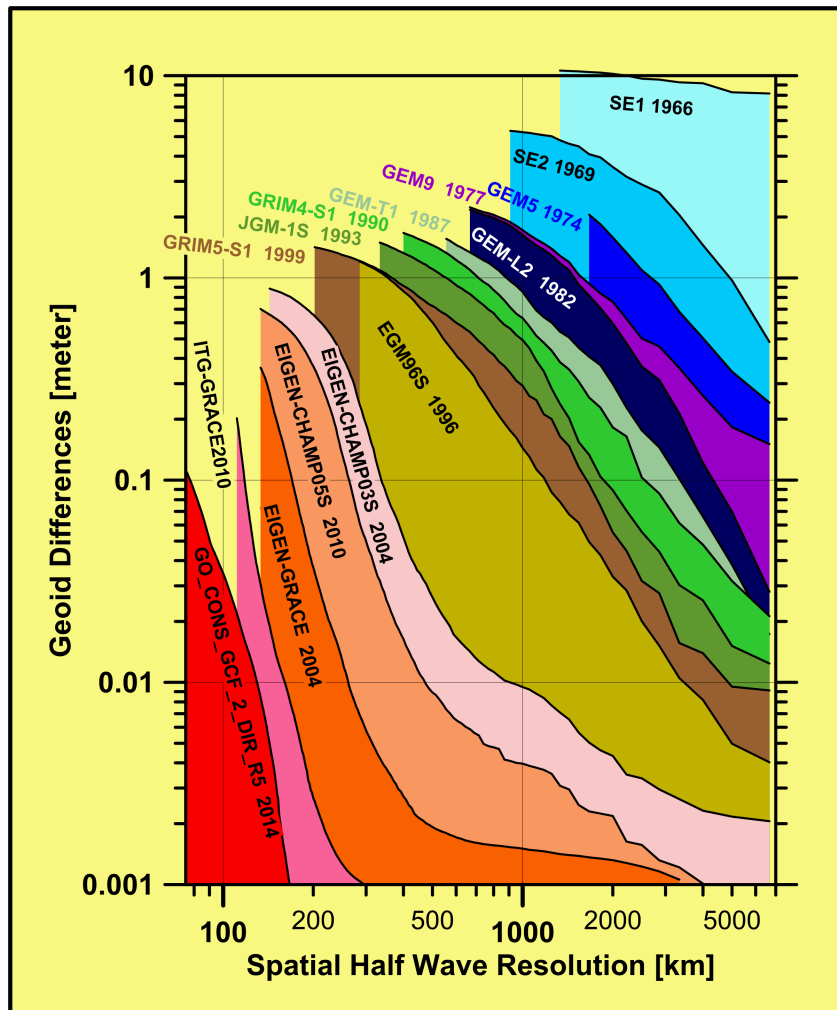


Figure 1: Geoid differences of satellite-only models to EIGEN-6C4 as a function of spatial resolution

functionals on the ICGEM web pages: <http://icgem.gfz-potsdam.de/ICGEM/> .

The improvement of the global models from the very start until today becomes clear from Figure 1. It shows geoid differences of some satellite-only models from the different eras to one of the recent combined models, EIGEN-6C4 (Förste et al, 2015), in dependence on the spatial resolution. This new combined model can be assumed as a good approximation of the reality (at least with respect to the older models), and the differences show how the accuracy and resolution of the models have been improved over the years. Whereas the first models had resolutions not better than 1000 km with differences to the new model of about 10 meters, the recent models based on GOCE data have a spatial resolution better than 100 km, and it seems that spatial details not smaller than 150 km extent are modeled with an accuracy better than 1 cm.

To visualize the gravity field, Fig. 2 shows geoid undulations of the model EIGEN-6C4, projected onto a sphere and exaggerated by a factor of 15000.

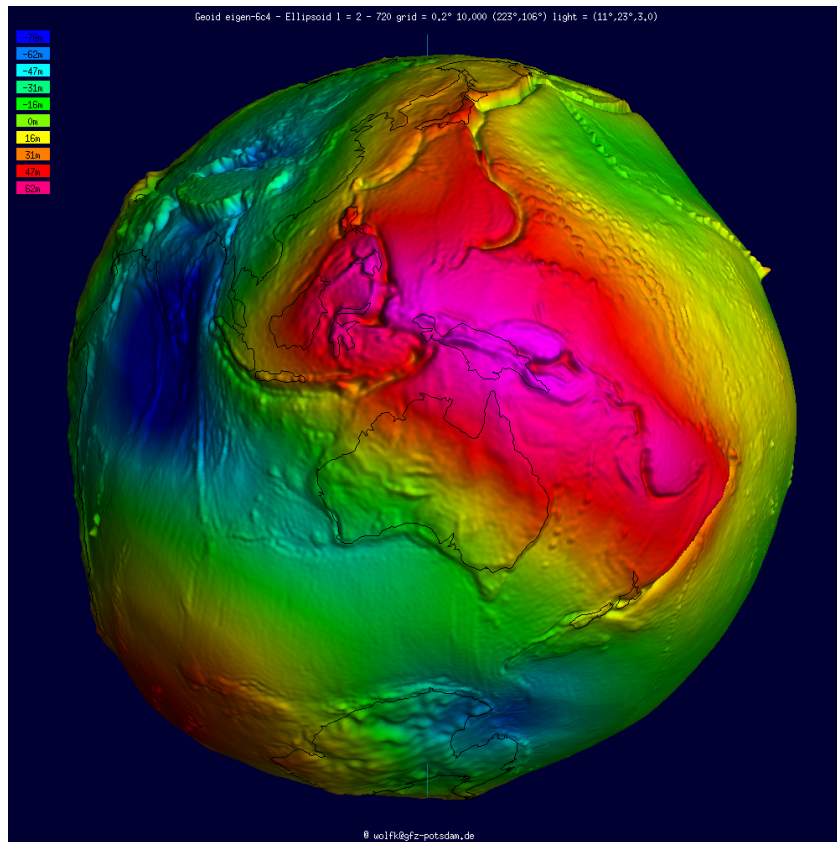


Figure 2: Geoid undulations of the model EIGEN-6C4, projected onto a sphere and exaggerated by a factor of 15000, color scale from -110 m (blue) to 88 m (magenta); the figure is a snapshot of the visualization service of ICGEM

4 Summary

In geodesy, a global model is an approximation of the gravity field of the Earth. It consists of the gravitational part according to Newton's law of attraction between masses and of the centrifugal part due to the rotation of the Earth. Such a model is a mathematical function which allows to compute different functionals of the gravity field (e.g., the gravity potential or the gravity vector) at all points on and outside the Earth. However, to compute geoid heights, gravity anomalies and gravity disturbances, which are of particular importance in geodesy, a defined reference system is necessary in addition to the global model. Usually the global models are represented as sets of spherical harmonic coefficients. Besides gravity measurements on the Earth's surface, since 1958 the information about the global Earth's gravitational field is mainly based on measurements made by artificial satellites.

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